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Level statistics of Anderson model of disordered systems: connection to Brownian ensembles

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Abstract

We find that the statistics of levels undergoing the metal–insulator transition in systems with Gaussian disorder and non-interacting electrons behaves in a way similar to that of the single parametric Brownian ensembles (Dyson 1962 *J. Math. Phys.* **3** 1191). The latter appear during a Poisson → Wigner–Dyson transition, driven by a random perturbation. The analogy provides analytical evidence for the single-parameter scaling of the level correlations in disordered systems as well as a tool to obtain them at the critical point for a wide range of disorders.

The spectral correlations of a disordered system are very sensitive to the behaviour of its eigenfunctions. The presence of disorder may cause localized waves in the system, implying lack of interaction between certain parts. This is reflected in the structure of the Hamiltonian matrix which is sparse in the site representation. The degree of sparsity of the matrix is governed by various system parameters e.g. dimensionality, shape, size and boundary conditions of the system. The variation of the disorder strength can lead to a metal–insulator transition (MIT), with eigenfunctions changing from a fully extended state (metal) to a strongly localized one (insulator) with partial localization in the critical region. The associated Hamiltonian also undergoes a transition, (*in effect only* due to variation of the relative strength of its elements), from a full matrix to a sparse or banded form and finally to a diagonal matrix. The statistical studies of levels for various degrees and types of disorders as well as system conditions require, therefore, analysis of different ensembles. Here the nature of the localization and its strength is reflected in the measure and the sparsity of the ensemble, respectively. Our objective in this paper is to obtain a mathematical formulation for the level correlations, common to a large class of system conditions (with Gaussian type randomness); the system information enters the formulation through a parameter, basically a function of various system parameters influencing the localization.

Recently it was shown that the eigenvalue distributions of various ensembles, with a multi-parametric Gaussian measure and independent matrix elements, appear as non-equilibrium stages of a Brownian type diffusion process [2]. Here the eigenvalues evolve with respect to

a single parameter which is a function of the distribution parameters of the ensemble. The parameter is therefore related to the complexity of the system represented by the ensemble and can be termed as the 'complexity' parameter. The solution of the diffusion equation for a given value of the complexity parameter gives the distribution of the eigenvalues, and thereby their correlations, for the corresponding system. A similar diffusion equation is known to govern the evolution of the eigenvalues of Brownian ensembles (BE) [1, 3] and many of its solutions for various initial conditions have already been obtained [4]. The analogy can then be used to obtain the level correlations for the Gaussian random matrix models of the disordered systems with non-interacting electrons. The presence of interactions introduces a correlation between matrix elements of the ensemble representing the system; the details of this case are discussed elsewhere [5].

The correlations in the single-electron spectra of disordered metals are governed by a variety of parameters, e.g. the associated energy ranges, degree of disorder, the dimensionality of the system etc. Here the two energy scales playing the dominant role are the Thouless energy E_c and the mean level spacing Δ. The E_c is given by the timescale needed by the wavepacket to diffuse through the sample. In the diffusive (metallic) regime and for energy scales δE smaller then E_c, the spectral correlations are well modelled by Wigner–Dyson (WD) ensembles [3]. For $\delta E > E_c$, the statistics deviate from the Wigner–Dyson case; however, the deviations are negligible for sample size $L \to \infty$. In the localized (insulator) phase too, the correlations are energy dependent but, in the limit $L \to \infty$, the levels are completely uncorrelated and their statistics can be modelled by the Poisson ensemble. However, the statistics in the critical region near the metal–insulator transition (Anderson type) is different from both Wigner–Dyson as well as Poisson statistics and depends on various system dependent features [7, 8]. Our study shows that the multi-parametric level statistics in the critical region can be well modelled by the single-parametric Brownian ensembles.

The paper is organized as follows. Section 1 contains a brief description of the simplest model of a disordered system using the independent electron approximation and the equation governing the evolution of its eigenvalues due to change of disorder etc. The properties of the BEs useful for the present study are given in section 2. Section 3 deals with the determination of the single parameter Λ governing the level statistics during MIT using the BE analogy. It also provides an explanation, in terms of Λ, of some of the observed features of the AE statistics. In section 4, the AE–BE analogy is used to obtain the analytical formulation of some of the unknown spectral fluctuations during MIT. Section 5 contains the details of the numerical comparison of the level statistics of the Anderson Hamiltonian with that of BEs and reconfirms our analytical results. The studies during the last decade indicate the surprising success of power law random banded matrices (1D system) as a model for Anderson ensembles [6]; as discussed in section 6, the success can be explained within the Λ formulation of the level statistics.

1. The multi-dimensional Anderson Hamiltonian

The Anderson model for a disordered system is described by a d-dimensional disordered lattice, of size L, with a Hamiltonian $H = \sum_n \epsilon_n a_n^+ a_n - \sum_{n \neq m} b_{mn}(a_n^+ a_m + a_n a_m^+)$ in the tight-binding approximation. The site energies ϵ_n, measured in units of the overlap integral between adjacent sites, correspond to the random potential. The hopping is assumed to connect only the z nearest neighbours (referred to by m) of each site. In the site representation, H turns out to be a sparse matrix of size $N = L^d$ with diagonal matrix elements as the site energies $H_{kk} = \epsilon_k$. The off-diagonals H_{kl} describe the interaction between two sites k and l; here, H_{kl} for two sites connected by hopping will be referred to as the hopping off-diagonal and

the rest as non-hopping off-diagonals. The level statistics of H can therefore be studied by analysing the properties of an ensemble of (i) sparse real symmetric matrices in the presence of a time-reversal symmetry and (ii) sparse complex Hermitian matrices in the absence of a time reversal.

We consider an ensemble of Anderson Hamiltonians (later referred to as an Anderson ensemble) with a Gaussian type disorder. The site energies $H_{kk} = \epsilon_k$ are thus independent Gaussian distributions $\rho_{kk}(H_{kk}) = \mathrm{e}^{-(H_{kk}-b_{kk})^2/2h_{kk}}$ with variance h_{kk} and mean b_{kk}. The hopping can be chosen to be isotropic or anisotropic, non-random or random (Gaussian). A general form of the probability density $\rho(H) \equiv \prod_{k,l;k\leqslant l} \rho_{kl}(H_{kl})$ of the ensemble, including all the above possibilities, can therefore be given by

$$\rho(H,h,b) = C \exp\left[-\sum_{s=1}^{\beta}\sum_{k\leqslant l}(1/2h_{kl;s})(H_{kl;s}-b_{kl;s})^2\right] \tag{1}$$

with subscript 's' of a variable referring to its components, β as their total number ($\beta = 1$ for the real variable, $\beta = 2$ for the complex one), C as the normalization constant, h as the set of the variances $h_{kl;s} = \langle H_{kl;s}^2\rangle$ and b as the set of all mean values $\langle H_{kl;s}\rangle = b_{kl;s}$. As is obvious, in the limit $h_{kl;1}, h_{kl;2} \to 0$, equation (1) corresponds to the non-random nature of H_{kl} (that is, $\rho_{kl}(H_{kl}) = \delta(H_{kl} - b_{kl})$). Note that although the non-hopping off-diagonals in Anderson matrix always remain zero the effective sparsity of the matrix changes due to change in the relative strength of the diagonals and the hopping off-diagonals. Thus, in the insulator limit (with almost no overlap between site energies due to strong disorder), the matrix behaves effectively as a diagonal one, the diagonals being very large as compared to hopping off-diagonals. In the opposite limit of very weak disorder when an average diagonal is nearly of the same strength as an average off-diagonal, the statistical behaviour of the matrix is the same as that of a matrix taken from a Wigner–Dyson ensemble [3]. The latter are the basis-invariant Gaussian ensembles of Hermitian type, with the same variance for almost all matrix elements. The statistical behaviour of levels in the Wigner–Dyson ensembles depends only on their symmetry class and is therefore universal in nature. The three main universality classes are described by a parameter β, basically a measure of the degree of level repulsion [3]:

(i) GOE with $\beta = 1$, corresponding to time-reversal symmetry and integer angular momentum,
(ii) GUE with $\beta = 2$ and no time-reversal symmetry,
(iii) GSE with $\beta = 4$ and time-reversal symmetry but half integer angular momentum.

A variation of disorder and hopping rate changes the distribution parameters of the probability density $\rho(H)$ and thereby its statistical properties. Using the Gaussian nature of ρ, it is easy to verify that under a change of parameters $h_{kl} \to h_{kl} + \delta h_{kl}$ and $b_{kl} \to b_{kl} + \delta b_{kl}$ the matrix elements H_{kl} undergo a diffusion dynamics along with a finite drift,

$$\sum_{k\leqslant l;s}\left[(2/\tilde{g}_{kl})x_{kl;s}\frac{\partial\rho}{\partial h_{kl;s}} - \gamma b_{kl;s}\frac{\partial\rho}{\partial b_{kl;s}}\right] = \sum_{kl;s}\frac{\partial}{\partial H_{kl;s}}\left[\frac{g_{kl}}{2}\frac{\partial}{\partial H_{kl;s}} + \gamma H_{kl;s}\right]\rho \tag{2}$$

where $x_{kl;s} \equiv 1 - \gamma\tilde{g}_{kl}h_{kl;s}$ with $\tilde{g}_{kl} = 2 - \delta_{kl}$ and $g_{kl} = 1 + \delta_{kl}$. γ is an arbitrary parameter, giving the variance of the matrix elements at the end of the evolution [2]. The above equation describes a multi-parametric flow of matrix elements from an arbitrary initial condition, say H_0. However, as discussed in [2], it is possible to define a 'complexity' parameter Y, a function of various distribution parameters $h_{kl;s}$ and $b_{kl;s}$, in terms of which the matrix elements undergo a single parametric diffusion,

$$\frac{\partial\rho}{\partial Y} = \sum_{kl;s}\frac{\partial}{\partial H_{kl;s}}\left[\frac{g_{kl}}{2}\frac{\partial}{\partial H_{kl;s}} + \gamma H_{kl;s}\right]\rho \tag{3}$$

with

$$Y = -\frac{1}{2M\gamma} \ln \left[\prod_{k \leqslant l}{}' \prod_{s=1}^{\beta} |x_{kl;s}||b_{kl;s}|^2 \right] + C \tag{4}$$

where $\prod'$ implies a product over non-zero $b_{kl;s}$ and $x_{kl;s}$. Further, C is a constant determined by the initial distribution, M is the number of all non-zero parameters $x_{kl;s}$ and $b_{kl;s}$ and $\beta = 1, 2$ for Hamiltonians with and without time-reversal, respectively.

The solution of equation (3) gives the state $\rho(H, Y|H_0, Y_0)$ of the flow at parameter Y, starting from an initial state H_0 with $Y = Y_0$. An integration over initial probability density result in the density given by equation (1) in terms of $Y(h, b)$: $\rho(H, Y) = \int \rho(H, Y|H_0, Y_0)\rho(H_0, y_0)\,\mathrm{d}H_0$. The evolution reaches a steady state when $\partial\rho/\partial Y \to 0$ with the ensemble $\rho(H)$ approaching the Wigner–Dyson limit, $\rho \propto \mathrm{e}^{-(\gamma/2)\mathrm{Tr}\,H^2}$.

As implied by equation (3), the variation of $\rho(H|H_0)$ depends on the changes of the parameters h_{kl}, b_{kl} (for all k, l) only through a function Y. This can be proved by considering a transformation of the M non-zero variables of the sets h and b to another set $\{Y, Y_2, \ldots, Y_M\}$ of M variables, $h_{kl} = h_{kl}(Y, Y_2, \ldots, Y_M)$ and $b_{kl} = b_{kl}(Y, Y_2, \ldots, Y_M)$. As shown in [2, 5], it is possible to define $Y, Y_2, \ldots, Y_M$ such that the $M - 1$ variables $Y_2, \ldots, Y_M$ remain constant during the evolution of ρ due to any change in sets h, b. The statistics during the transition is therefore governed by Y only. The choice of the $Y_2, \ldots, Y_M$ depends on the system under consideration. For a transition preserving the lattice structure, these constants turn out to be functions of the site indices in the lattice. For example, the variances h_{kl} in the Anderson ensemble are functions of the disorder as well as the site indices k, l; Y can then be identified as a function of disorder while Y_j ($j > 1$) as functions of site indices. Further, as these constants do not appear explicitly in equation (3), its solution and therefore the ensemble statistics is independent of the specific values of the constants.

The flow described by equation (3) can start from any initial state; the only constraint on the choice is that the parameters Y_j, $j > 2$, for the initial ensemble should be same as those for the ensemble $\rho(H, h, b)$. As shown below by an example, the initial state can also be chosen as the insulator limit of the disordered system, described by an ensemble of diagonal matrices. Although this corresponds to the same value for all initial off-diagonal variances (that is, zero), a choice of different rates of change of h_{kl} with respect to Y can result in different possible values for each h_{kl} at a later stage.

As an example, consider an Anderson system with a Gaussian site disorder (of variance $W^2/12$ and mean zero), the same for each site, and an isotropic Gaussian hopping with a non-random component (of variance $W_s^2/12$ and mean t_s with $s = 1, 2$ for real and imaginary parts respectively) between nearest neighbours (referred to as ensemble **G** later on). The corresponding probability density can be described by equation (1) with

$$h_{kk} = W^2/12, \qquad b_{kk} = 0$$
$$h_{kl;s} = f_1(kl; s)\ W_s^2/12, \qquad b_{kl;s} = f_2(kl; s)\ t_s$$

where $f_1(kl; s) = 1$, $f_2(kl; s) = 1$ for for $\{k, l\}$ pairs representing hopping, $f_1(kl; s) \to 0$ and $f_2(kl; s) \to 0$ for all $\{k, l\}$ values corresponding to disconnected sites. As is obvious, here the distribution parameters h_{kl} depend on more than one system parameter, namely, the disorder parameters W, W_1 and W_2 as well as various functions of site indices. The latter, being invariant with motion, give the parameters $Y_2, \ldots, Y_M$. The Y for this case can be obtained by using equation (4),

$$Y = -\frac{N}{2M\gamma}\alpha + C \tag{5}$$

$$\alpha = \ln|1 - \gamma W^2/12| + (z/2)\sum_s \ln\left[|1 - \gamma W_s^2/6||t_s + \delta_{t_s 0}|\right] + C, \tag{6}$$

with $M = \beta N(N+z(1-\delta_{t0})+2-\beta)/2 \approx \beta N^{2+\epsilon}/2$. Here zN is the number of connected sites (nearest neighbours) which depends on the topology and the dimensionality d of the system and ϵ is a function of z, $\epsilon(z) = (\log(N + z(1 - \delta_{t0}) + 2 - \beta)/\log N) - 1; \epsilon \to 0$ for $z \ll N$.

Now consider an insulator as the initial state (in the same site basis as used for $\mathbf{G}$) with zero hopping, that is, $W_s = 0, t_s = 0$ and a Gaussian site disorder with variance $(W^2/12) = (2\gamma)^{-1}$ (referred to as $\mathbf{G}_0$). This corresponds to an ensemble of diagonal matrices with $h_{kk} = (2\gamma)^{-1}$, $h_{kl;s} = 0$ for $k \neq l$ and $b_{kl;s} = 0$ for all k, l. A substitution of these values in equation (5) gives the initial value of Y, say Y_0, where $Y_0 = -\frac{N}{2\gamma M}\alpha_0 + C$ with $\alpha_0 = -\ln 2$. Note that the basis being the same, the parameters Y_j (for $j \geqslant 2$) are the same for both $\mathbf{G}$ and $\mathbf{G}_0$. (The advantage of choosing the above initial state is explained later.) As is obvious, starting from $\mathbf{G}_0$, a variation of diagonal disorder W, hopping parameters W_s and t_s with rates

$$\frac{\delta h_{kk}}{\delta W} = W/6,$$
$$\frac{\delta h_{kl;s}}{\delta W_s} = W_s f_1(kl;s)/6, \qquad \frac{\delta b_{kl;s}}{\delta t_s} = t_s f_2(kl;s) \qquad k \neq l$$

can lead to the ensemble G. Using $(\partial Y_j/\partial x) = 0$ for $j \geqslant 2$ with $x \equiv W, W_s, t_s$, and equation (5) to obtain $(\partial Y/\partial x)$, it can be seen that the above rates correspond to

$$\frac{\partial h_{kk}}{\partial Y} \propto |1 - \gamma W^2/12|,$$
$$\frac{\partial h_{kl}}{\partial Y} \propto f_1|1 - \gamma W_s^2/6|, \tag{7}$$
$$\frac{\partial b_{kl}}{\partial Y} \propto f_2 t_s;$$

the variances and means of different matrix elements therefore change with different rates with Y.

The distribution P of the eigenvalues E_n for a metal (for the energy ranges with fully extended eigenfunctions) is given by the Wigner–Dyson distribution, $P(\{E_n\}) = \prod_{i<j}|E_i - E_j|^\beta \mathrm{e}^{-\frac{\gamma}{2}\sum_k E_k^2}$, and that for an insulator by a Poisson distribution [7]. The distribution for various transition stages can be obtained by integrating ρ over the associated eigenvector space. Let $P(\{E_n\}, Y(h,b))$ be the joint probability of finding eigenvalues λ_i of H between E_i and $E_i + \mathrm{d}E_i$ $(i = 1, 2, \ldots, N)$ at a given h and b; it can then be expressed as $P(\{E_n\}, Y) = \int \prod_{i=1}^N \delta(E_i - \lambda_i)\rho(H, Y)\,\mathrm{d}H$. Using the above definition in equation (3), it can be shown that the eigenvalues of $\rho(H)$ undergo a diffusion dynamics along with a finite drift due to their mutual repulsion (see [2] also),

$$\frac{\partial P}{\partial Y} = \sum_n \frac{\partial}{\partial E_n}\left[\frac{\partial}{\partial E_n} + \sum_{m\neq n}\frac{\beta}{E_m - E_n} + \gamma E_n\right]P. \tag{8}$$

Again, the steady state of the evolution is given by the limit $\partial P/\partial Y \to 0$; $P(\{E_n\})$ in this limit turns out to be a Wigner–Dyson distribution.

Equation (8) can be used to obtain the correlations between levels. For example, a knowledge of its solution P gives the static correlations

$$R_n(E_1, E_2, \ldots, E_n; Y) = \frac{N!}{(N-n)!}\int P(\{E_j\}, Y)\,\mathrm{d}E_{n+1}\ldots\mathrm{d}E_N. \tag{9}$$

P can be obtained by using the analogy of equation (8) with the equation governing the evolution of the eigenvalues of Brownian ensembles (BEs) of Hermitian type [1, 3]. The latter

has been studied in great detail in the past and many of its statistical spectral properties are already known [4]. A brief description of the BE is given in the next section.

It should be noted here that the single-parametric evolution of the matrix elements of the AE in terms of the complexity parameter $Y - Y_0$ would result in a similar evolution for their eigenvector components too; this can be shown by integrating equation (3) over all eigenvalues. However, in this paper we confine ourselves to the discussion of eigenvalue statistics only; the details for the eigenvector statistics will be published elsewhere.

2. Spectral properties of Brownian ensembles

The stationary random-matrix ensembles were introduced in the past to model quantum mechanical operators of complex systems in which a certain set of quantities (for example, total spin, charge or isotopic spin) was exactly conserved; no other integral of the motion existed even approximately [1, 3]. The total set of the states of the system could then be divided into subsets, each subset corresponding to a particular set of values for the conserved quantities. This divides the matrix representation of the operator into various blocks; the deterministic uncertainty due to complicated nature of the interactions leads to randomization of the blocks. Due to lack of correlation between energy levels of states belonging to different subsets, different blocks are uncorrelated. The statistics of the levels within one subset can then be described by a separate random matrix model which can be of various types based on the underlying symmetry [1, 3].

The stationary ensembles are inappropriate models for systems possessing approximate conservation laws. However, such systems occur more frequently in practice, which motivated Dyson to introduce the Brownian ensembles (BEs) of random matrices [1, 3]. As the latter have been discussed in detail in the past e.g. in [3, 4] (and references therein), here we give only a brief review of the BEs related to Hermitian matrices. Consider the Hamiltonian operator H of a system with its elements given by H_{kl} at 'time' λ and $H_{kl} + \delta H_{kl}$ at 'time' $\lambda + \delta\lambda$. A Brownian motion of H is defined by requiring that each δH_{kl} is a random variable with the moments $\langle \delta H_{kl;s} \rangle = -\gamma H_{kl;s} \delta\lambda$, $\langle (\delta H_{kl;s})^2 \rangle = g_{kl} \delta\lambda$ [3]. The evolution of the distribution of matrix elements, from any arbitrary initial state, can then be given by a Fokker–Planck equation which has the same form as equation (3) with $Y \propto \lambda^2$. For $\lambda \to \infty$, the distribution approaches steady state which corresponds to one of the stationary ensembles. The crossover to stationarity is rapid and discontinuous, as a function of λ, for infinite matrix sizes or very large energy ranges.

A Brownian ensemble can therefore be described as a non-stationary state of the matrix elements undergoing a crossover due to a random perturbation of a stationary ensemble by another one. For example, in the case of Hermitian operators, a Brownian ensemble H can be given as $H = \sqrt{f}(H_0 + \lambda V)$ (with $f = (1 - \lambda^2)^{-1}$); here V is a random perturbation of strength λ, taken from a stationary ensemble, and applied to an initial stationary state H_0 (see also [4]). Using second order perturbation theory, it can be shown that the eigenvalues E_j, $j = 1, 2, \ldots, N$ of H execute a Brownian motion too, with their evolution described by an equation the same as equation (8) (with $Y \propto \lambda^2 f$). The eigenvalue statistics (e.g. static correlations given by equation (9)) of a BE can then be obtained by solving equation (8). Equation (8) is equivalent, under a Wick rotation, to the Schrödinger equation of the Calogero–Sutherland Hamiltonian; the equivalence has been used to obtain the eigenvalue correlations for many BEs [4]. It is shown moreover that the crossover in correlations is governed, for small λ and large N, by a rescaled parameter Λ which measures locally the mean-square symmetry breaking matrix element in units of the mean eigenvalue spacing of H.

The type of BE appearing during the crossover depends on the nature of stationary ensembles H_0, V and their different pairs may give rise to different BEs [4]. The present

knowledge of ten types of stationary ensembles [3] leads to the possibility of many such crossovers and, consequently, many types of BEs. For example, the Hamiltonian of a disordered system or an autonomous chaotic system, with time-reversal symmetry, can usually be modelled by a Gaussian orthogonal ensemble. The breaking of time-reversal symmetry e.g. by switching of a magnetic field, with λ as a measure of the breaking, perturbs the Hamiltonian H_0. The statistical behaviour of the system now depends on the energy range of interest. At asymptotically large energies, the statistics can be modelled by Gaussian unitary ensembles; however, at intermediate energies with sufficiently small values of λ, an intermediate statistics (a BE between GOE and GUE) would be obtained, indicative of a non-equilibrium behaviour. Similarly, if the system is integrable with regular classical motion for $\lambda = 0$ and fully chaotic for $\lambda \neq 0$, the statistics undergoes the Poisson $\rightarrow$ GOE crossover (the BE in this case is a superposition of Poisson and GOE ensembles). For many types of crossovers, beginning from various stationary states e.g. GOE, GSE, 2GOE, Poisson, uniform etc and approaching GUE in the limit $\lambda \rightarrow \infty$, the second order correlation functions for all Λ have been explicitly evaluated [4]; for the other transitions the correlations are given implicitly by a hierarchic set of relations [4, 2].

Here we discuss only the BEs appearing during a transition from the Poisson to the Wigner–Dyson ensemble (referred to as a Wigner–Dyson transition or WDT) caused by a perturbation of the former by the latter (that is, taking H_0, V as the Poisson and the Wigner–Dyson ensemble respectively). As this transition results in a change of localized eigenstates to delocalized ones, its relevance for the study of MIT is intuitively suggested. The BEs related to the Poisson $\rightarrow$ Wigner–Dyson transition can be described by an $N \times N$ ensemble H represented by the following probability distribution for all (independent) matrix elements:

$$\rho(H) \propto \exp\left[-\gamma \sum_{i=1}^{N} H_{ii}^2 - 2\gamma(1+\mu) \sum_{i<j} |H_{ij}|^2\right] \tag{10}$$

with $(1+\mu) = (\lambda^2 f)^{-1}$; here $H = H_0$ for $\lambda \rightarrow 0$ or $\mu \rightarrow \infty$. An ensemble H given by the above measure is also known as the Rosenzweig–Porter ensemble (RPE); note that it also corresponds to an ensemble of Anderson Hamiltonians with very long range, isotropic, random hopping.

Equation (8) describes the evolution of the eigenvalues of a generalized Gaussian ensemble with a probability density (1) and is therefore applicable for the BEs defined by a probability density (9) too. A comparison of measure (9) with measure (1), gives a variance $h_{kl;s} = (4\gamma(1+\mu))^{-1}$, $h_{kk;s} = (2\gamma)^{-1}$ and mean $b_{kl;s} = 0$ for all k, l and s indices. Using these values in equation (4), the parameter Y for the BE case can be given as

$$\begin{aligned} Y &= -\frac{1}{2\gamma}\frac{(N-1)}{(N+2-\beta)} \log\left(1 - \frac{1}{2(1+\mu)}\right) + Y_0 \\ &\approx \frac{1}{4\gamma\mu} + Y_0 \qquad (\text{for } \mu \gg 1) \end{aligned} \tag{11}$$

with $M = \beta N(N+2-\beta)/2$ and $Y_0 = \frac{N}{2\gamma M} \ln 2 + C$ as the complexity parameter of the ensemble H_0 (note that $Y = Y_0$ for $\mu \rightarrow \infty$).

A typical matrix in ensemble (9) has the diagonal elements of order $\gamma^{-1/2}$ and off-diagonals of the order of $(\gamma\mu)^{-1/2} (= o(Y - Y_0)^{1/2})$. The number of off-diagonals being N times more than the diagonals, the matrix behaviour is governed by the parameter μ. Thus, for large BE ($N \rightarrow \infty$), a radical change from the Wigner–Dyson case can only occur if μ increases more rapidly than N (which makes the total strength of the off-diagonals weaker than that of the

diagonals). This results in three different regimes of the mean-level density $R_1(E)$ [9]:

$$R_1(E) = \frac{N}{\sqrt{\pi}} e^{-E^2} \qquad \text{for } N(Y - Y_0) \to 0 \tag{12}$$

$$= \frac{\sqrt{8N\gamma(Y - Y_0) - E^2}}{4\gamma\pi(Y - Y_0)} \qquad \text{for } N(Y - Y_0) \to \infty \tag{13}$$

$$= NF(E, a) \qquad \text{for } N(Y - Y_0) = a \tag{14}$$

with a as an N-independent constant. Although the exact form of the function $F(E, a)$ is not known, its limiting behaviour can be given as follows: $F(E, a) \approx e^{-E^2}/\sqrt{\pi}$ for $a \ll 1$ and $F(E, a) \approx (4\pi\gamma a)^{-1}\sqrt{8\gamma a - E^2}$ for $a \gg 1$, $E^2 \ll a$ [9].

The first order correlation R_1, also known as the mean level density, changes from an exponential to semi-circular form at the scale of $(Y - Y_0) \sim N\Delta_1^2$ with Δ_1 as the local mean level spacing; the evolution of R_1 can therefore be described in terms of the parameter $(Y - Y_0)$. However, the transition of higher order correlations R_n $(n > 1)$ occurs at a scale determined by $(Y - Y_0) \sim \Delta_1^2$ [4, 9]. As a result, their transition to equilibrium, with $|Y - Y_0|$ as the evolution parameter, is rapid and discontinuous for infinite dimensions of matrices [1]. But for small Y and large N, a smooth crossover can be seen in terms of a rescaled parameter $\Lambda(E)$:

$$\Lambda(E, Y) = |Y - Y_0|/\Delta_1^2. \tag{15}$$

For finite N, Λ varies smoothly with changing μ: $\Lambda = R_1^2/4\gamma\mu$. This results in a continuous family of BEs, parametrized by Λ, existing between the Poisson and the Wigner–Dyson limit. However, the level statistics for the large BE $(N \to \infty)$ can be divided into three regions [9].

(i) *Poisson regime:* $N^2(Y - Y_0) \to 0$. The off-diagonals, responsible for the correlation between levels, are negligible. The lack of repulsion between levels results in a mean level spacing $\Delta_1 \propto N^{-1}$ (see equation (12)), thereby giving $\Lambda \to 0$ and the Poisson statistics.

(ii) *WD regime:* $N^2(Y - Y_0) \to \infty$. The contribution from both the diagonals as well as off-diagonals is of the same order, leading to long range correlations between levels. The repulsion of levels now results in a mean level spacing $\Delta_1 \propto N^{-1/2}$ (see equation (13)) which gives $\Lambda \to \infty$ and Wigner–Dyson statistics.

(iii) *Critical regime:* $N^2(Y - Y_0) = (4\gamma c)^{-1} =$ a constant. For $\mu = cN^2$ with c as a constant independent of N, a sequence of approximately $o(1/\sqrt{c})$ levels show Wigner–Dyson behaviour. The more distant levels display weak correlations of the type existing near the Poisson limit resulting in a $\Delta_1 \approx o(1/N)$. The parameter Λ is therefore N independent:

$$\Lambda(E) = (1/4c\pi\gamma)e^{-E^2}; \tag{16}$$

note it is also independent of the symmetry parameter β.

The finite, non-zero Λ value for $\mu = cN^2$ in the limit $N \to \infty$ therefore gives rise to a third statistics, intermediate between the Poisson and the Wigner–Dyson ensemble, which is known as the critical Brownian ensemble (CBE). This being the case for arbitrary values of c (non-zero and finite), an infinite family of critical BEs, characterized by c (or $\mu_c = cN^2$), occurs during WDT. Note that the critical BEs, with $c \to \infty$ and $c \to 0$, correspond to the Poisson and the Wigner–Dyson limit, respectively.

The presence of a family of the critical BEs can be seen from any of the fluctuation measures for WDT. One traditionally used measure in this regard is the relative behaviour of the tail of the nearest neighbour spacing distribution $P(s, \Lambda)$, defined as $\alpha(\delta, \Lambda) = \int_0^\delta (P(s, \Lambda) - P_w(s))\,ds / \int_0^\delta (P_p(s) - P_w(s))\,ds$ with δ as any one of the crossing points of $P_w(s)$ and $P_p(s)$ (here subscripts w and p refer to the Wigner–Dyson case and the Poisson

case respectively) [10]. In the limit $N \to \infty$, $\alpha = 0$ and 1 for the Wigner–Dyson and the Poisson limit respectively. The figure 1 shows the numerically obtained behaviour of α (for $\delta \approx 2.02$) with respect to $|z - c|$ $(=|\mu - \mu_c|N^{-2})$ for a fixed c (arbitrarily chosen) with z as a variable; here z and c are the values of the parameter μN^{-2} for a general BE and a critical BE respectively. The constant value of α at $|z - c| = 0$ for different N values confirms the size independence of the level statistics of BE with parameter $\mu = cN^2$ and therefore its critical nature; we have verified it for other c values too and find, for finite, non-zero c values, $0 < \alpha < 1$. Further, the convergence of α values for BEs with different μ- and N values on two branches indicates the presence of a scaling behaviour in the level statistics of BEs with $|z - c|$ $(=|\mu - \mu_c|N^{-2})$ as the scaling parameter.

As shown in figure 1(a), α for a critical BE is between zero and unity. A fractional value of α indicates a tail behaviour of the critical BE different from that of the Poisson as well as the Wigner–Dyson limit. As shown in figure 1(b), the $P(s)$ for a critical BE with a finite parameter c has an exponential tail, $P(\text{large } s) \sim \exp(-\kappa s)$; this behaviour of $P(s)$ is also sometimes referred to as a semi-Poisson distribution, due to the presence of repulsion at small energy scales and exponential decay at large separations.

3. Analogy between Brownian ensembles and Anderson ensembles

The same evolution equations of P for AEs and BEs imply a similarity in their eigenvalue distributions for all Y values, under similar initial conditions (that is, $P(\mu, Y_0)$ the same for both cases). As a result, one obtains the analogous evolution equations for their correlations R_n too. The mean level density $R_1(E, Y - Y_0)$ of an AE can therefore be given by the level density of a BE with the same $|Y - Y_0|$ value (and appearing during a Poisson to Wigner–Dyson transition). Similarly, the analogy of evolutions of R_n $(n > 1)$ in the two cases implies

(i) a smooth crossover of R_n for finite size Anderson systems in arbitrary dimensions,
(ii) the parameter Λ governing the smooth crossover of R_n for finite size AEs can again be defined by equation (15), with $Y - Y_0$ given by equation (4) and Δ_1 as the local mean level spacing for the AEs,[1]
(iii) the correlations R_n, $n > 1$, of an AE can be given by those for a BE with the same Λ value although their parameters Y (as well as level densities) may be different,
(iv) the discontinuity of the transition of R_n for infinite size of the Anderson matrix,
(v) the existence of a size-invariant level statistics, different from the two end-points, if an AE has $\Lambda =$ size independent; the statistics survives the thermodynamic limit $L \to \infty$. As explained later by an example, the above condition on Λ is satisfied at the critical point of $d > 2$-dimensional Anderson systems; the corresponding level statistics is referred to as critical.

Implications (i) and (iv) are in good agreement with known results about AE correlations [11]. Implications (ii), (iii) and (v) indicate the single-parametric dependence of the level statistics for AEs. The parameter Λ for the AE and BE will henceforth be referred to as Λ_a and Λ_b, respectively. The level statistics of a finite-size AE at Λ_a is then given by a BE with its parameter μ satisfying the condition

$$\Lambda_a = \Lambda_b \tag{17}$$

[1] The higher order correlations basically being the measures of the fluctuations of the density around its average value, their comparison in two different spectra makes sense only if the fluctuations are measured with respect to the same background. This requires an unfolding of each spectrum, that is, rescaling by its mean level density before comparison with another spectrum [4]. As the parameter governing the evolution in the rescaled spectrum is Λ, the higher order correlations of an AE are given by a BE with the same Λ value.

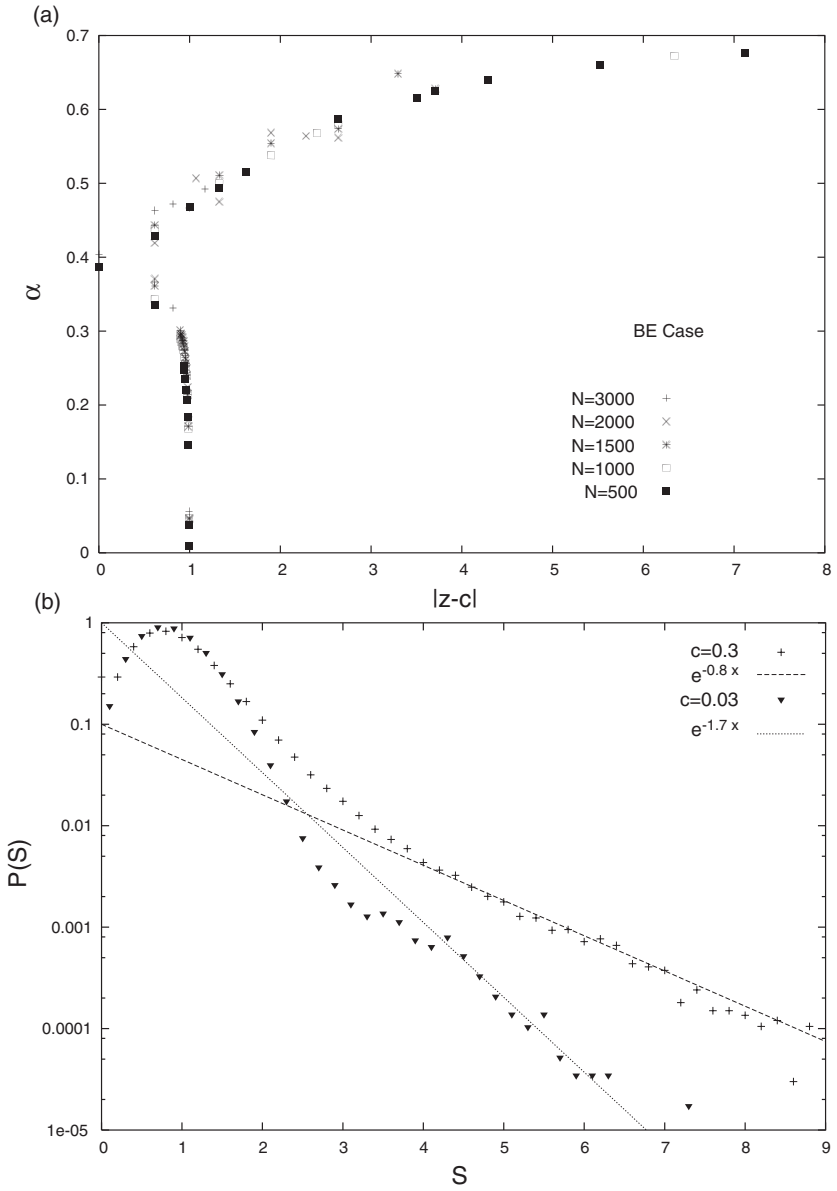

Figure 1. The study of critical BEs: (a) the scaling behaviour of the integrated nearest neighbour spacing distribution α during WDT. Note that α values for BEs with different parameters μ and sizes N converge on the same two curves, thus indicating the α dependence on a specific combination of μ and N, namely, $z = \mu/N^2$. Further, at $z = c$, α remains unchanged for different N values, thus indicating a critical point of BEs. (b) The comparison of the tail of the $P(S)$ distribution for two of the critical BEs with function $e^{-\kappa s}$ for the $\beta = 2$ case. For intermediate c-ranges, the $P(s)$ is well fitted by the function $e^{-\kappa s}$. The fitting, however, seems to be poor for smaller c values which is as expected, due to the statistics approaching the GUE limit (which corresponds to $P(s) \sim e^{-\pi s^2/4}$). The fitted κ values are as follows: (i) $\kappa = 0.8$ for $c = 0.3$, (ii) $\kappa = 1.7$ for $c = 0.03$. The above κ values seem to deviate significantly from the relation $\kappa = (\pi/4\gamma c)$ (obtained by using $\kappa = (2\chi)^{-1}$ with χ given by equation (35) and $\gamma = 2$). We have seen a similar deviation for the BEs with $\beta = 1$ too. This suggests the non-validity of relation $\kappa = (2\chi)^{-1}$ in general although it seems to work for some c values (see for example figure 3).

where $\Lambda_b = R_1^2/4\gamma\mu$ with R_1 as the level density of the BE; the determination of Λ_a is explained later by using an example. As the BEs with different combinations of the parameters μ and N can have the same Λ_b, the correlations of a finite size AE can be mapped to many BEs. However, the critical BE corresponding to a critical AE is unique; this can be understood as follows. Using equations (16) in (17), the parameter c for a critical BE corresponding to an AE can be given by

$$c = (4\pi\gamma\Lambda_a)^{-1}\mathrm{e}^{-E^2}. \tag{18}$$

The Λ_a for a critical AE being size independent, its critical BE analogue remains the same for all system sizes. However, the Λ_a for an AE, away from its critical point, is size dependent and therefore corresponds to different c values (that is, different critical BEs) for different system sizes.

The Λ_a for a disordered system can be determined by a knowledge of $Y - Y_0$ and Δ_1. The complexity parameter $Y - Y_0$ is system specific and depends on various system parameters. For a d-dimensional disordered system of linear size L, the local mean level spacing Δ_1 within a correlation volume of linear dimension ζ is related to the mean level density R_1: $\Delta_1 = (L/\zeta)^d R_1^{-1}$ where ζ is the localization length or correlation length in the case of localized states and extended states, respectively[2] [11]. The ζ can be determined by a knowledge of the wavefunction correlations, e.g. inverse participation ratio I_2 [15]: $\zeta^d \propto (I_2)^{-1}$ for localized eigenstates [15]. As mentioned in the last paragraph of section 1, the wavefunction statistics, and therefore ζ, can also be described, in principle, by a complexity parameter formulation. However, the related work being still in progress, we use, in this paper, the ζ results given by previous studies.

Let us consider the example **G** given in section 2; its parameter Y is given by equation (5). The initial state $\mathbf{G}_0$ has parameter Y_0 the same as that of the initial state chosen in the BE case in section 2. Note that as $N|Y - Y_0| \to$ an N-independent function for the case **G**, its R_1 is given by equation (14) with $R_1 = NF$. Using equations (5) in (15), Λ for the case **G** can be given as

$$\Lambda_a(E, Y) = \left(\frac{|\alpha - \alpha_0|F^2}{\beta\gamma}\right)\zeta^{2d}L^{-d} \tag{19}$$

with $F(E)$ giving the energy dependence of Λ (as $|\epsilon| \approx 0$ for large N). Following equation (17), the level statistics at $\Lambda_a \to 0, \infty$ corresponds to Poisson (or insulator limit) and Wigner–Dyson behaviour (metallic limit), respectively. In finite systems, a change of disorder results in a smooth variation of ζ as well as $\alpha - \alpha_0$ and, therefore, Λ_a which induces a crossover of the level statistics from the Poisson to the Wigner–Dyson ensemble. The intermediate level statistics at each Λ_a of a finite size AE is then given by an $N_1 \times N_1$ BE with its parameter μ satisfying the relation $\Lambda_a = \Lambda_b$:

$$\mu \approx \beta(4\pi|\alpha - \alpha_0|F^2)^{-1}\zeta^{-2d}L^d R_1^2 \tag{20}$$

with $R_1 \equiv R_1(E; \mu, N_1)$ as the level density of the BE. As is obvious, the determination of μ from the above equation is not easy, both sides being μ dependent. The R_1 for the critical BEs being μ independent (given by equation (12)), it is preferable to map an AE to a critical BE; the substitution of equation (19) in (18) gives the c parameter for the corresponding critical BE:

$$c \approx \beta(4\pi|\alpha - \alpha_0|F^2\mathrm{e}^{E^2})^{-1}\zeta^{-2d}L^d. \tag{21}$$

[2] For a d-dimensional disordered system, the number of states in a volume of linear dimension ζ in d dimensions is $n(0)\zeta^d$, with $n(0)$ as the density of states at the Fermi level and ζ as the localization length. Consequently, the typical energy separation between such states is $\Delta_1(E, Y) = (n(0)\zeta^d)^{-1}$. Similarly, the mean level spacing of states in the full length of the spectrum is $\Delta(E, Y) = (n(0)L^d)^{-1}$, which gives $R_1 = \Delta^{-1} = n(0)L^d$. For disordered systems, Δ_1 can therefore be expressed in terms of the mean level density R_1: $\Delta_1 = (L/\zeta)^d R_1^{-1}$.

Thus each state of disorder in an AE of size L ($N = L^d$) can be mapped to a critical BE with the parameter c given by equation (21). Note that, the right side of equation (21) being energy dependent, different energy ranges of a given AE will, in general, correspond to different critical BEs.

Equation (19) indicates the sensitivity of the parameter Λ_a to localization length ζ and system size L. It is now well known that ζ is a function of disorder strength, energy and system size L as well as the dimensionality of the system. For systems with finite L (in arbitrary dimensions $d \geqslant 1$), ζ, at a fixed energy, decreases with increasing disorder strength. Consequently, in the strong disorder limit (where $\zeta \sim o(L^0)$), $\Lambda_a \to 0$ and the level statistics of the AE approaches Poisson behaviour (as $\Lambda_b \to 0$ for its BE analogue). In the opposite limit $\zeta \sim o(L)$ of weak disorder, $\Lambda_a \to \infty$ and, therefore, the statistics of the AE is given by a BE at $\Lambda_b \to \infty$ which corresponds to Wigner–Dyson behaviour. By a suitable choice of disorder, however, it is possible to achieve finite values of the ratio ζ^2/L (due to finite L) in arbitrary dimensions which in turn gives finite, non-zero Λ_a and, thus, a finite c for its BE analogue. The latter implies that the AE statistics is intermediate between the Poisson and the Wigner–Dyson limit, with an exponential decay of the tail of its nearest neighbour spacing distribution $P(s)$. For finite L, therefore, a smooth crossover from Poisson to Wigner–Dyson statistics can be seen, *for any dimensionality* $d \geqslant 1$, as a function of Λ_a by varying the disorder strength. Note that two finite size AEs of different dimensions can show the same level statistics if their parameters Λ_a are equal. For example, consider the behaviour of levels of a one-dimensional AE of size L and at a disorder strength which gives $Y = Y_1$. The behaviour will be the same as that of a three-dimensional AE of linear size L, at a disorder strength which gives $Y = Y_3$ where $Y_3 = Y_1(\zeta_1 R_1^{(1)}/\zeta_3 R_1^{(3)})^2$; here $R_1^{(d)}$ and ζ_d refer to the mean level density and localization length in dimension d, respectively.

The dimensionality dependence of the Anderson transition and the critical level statistics is well known. For example, the level statistics at the critical disorder for $d > 2$-dimensional, finite systems shows a 'semi-Poisson' behaviour which survives the infinite size limit. The same behaviour is seen for $d \leqslant 2$-dimensional finite systems, in a regime where $\zeta \sim L$; however, the statistics approaches a Poisson behaviour in the thermodynamic limit. The above behaviour can be explained within the 'Λ formulation'. As mentioned in section 2, a 'semi-Poisson' behaviour of the level statistics is a characteristic of critical BEs with finite c parameters and therefore of the AEs with finite Λ_a parameters (see equation (19)). The AE statistics is expected to maintain its semi-Poisson behaviour even in the thermodynamic limit if $\Lambda_a =$ size independent. In this sense, Λ_a can be identified with the dimensionless conductance g: both g, $\Lambda_a \to 0, \infty$, constant correspond to the same statistical limit, namely, Poisson, Wigner–Dyson and a critical level statistics, respectively. In fact, Λ_a can be expressed in terms of the dimensionless conductance g of the system. This is because g is connected to ζ (based on scaling theory of localization for disordered systems [12]): $\zeta \propto L \log g^{-1}$ for exponentially localized states, $\zeta \propto |(g/g_c) - 1|^{-\nu}$ near the critical point and $\zeta \sim o(L)$ with $g \propto L^{d-2}$. For example, using equation (19), the Λ_a–g relation for case G near the critical point can be given as

$$\Lambda_a(E, Y) = \left(\frac{|\alpha - \alpha_0| F^2 \zeta_0^{2d}}{\beta\gamma}\right) |(g/g_c) - 1|^{-2\nu d} L^{-d} \tag{22}$$

with g_c as the critical point conductance and ν as the critical exponent.

As indicated by equation (19), the size independence of Λ_a is governed by the size dependence of the localization length. For example, for a d-dimensional disordered system,

with $|Y - Y_0| \simeq o(L^{x_1 d})$, the level density is $R_1(Y - Y_0) \simeq o(L^{x_2 d})$ where

$$x_2 = \begin{cases} 1 & \text{for } x_1 \leqslant -1, \\ (1 - x_1)/2 & \text{for } x_1 \geqslant -1. \end{cases} \tag{23}$$

Thus the critical point of the level statistics (that is, $\Lambda =$ size independent) can exist only if, in the thermodynamic limit, the disorder conditions in the system give rise to a localization length $\zeta \sim o(L^{x_3})$ where $x_3 \approx (2 - x_1 - 2x_2)/2$ or, equivalently,

$$x_3 = \begin{cases} |x_1|/2 & \text{for } x_1 \leqslant -1 \\ 1/2 & \text{for } x_1 > -1. \end{cases} \tag{24}$$

The existence or non-existence of a critical level statistics in an AE therefore depends on the size dependence of ζ, which in turn is sensitive to the dimensionality of the system [12]:

Case $d \leqslant 2$. For a $d = 1$ disordered lattice, almost all states are known to be exponentially localized even in the weak disorder limit. ζ in this case is finite, $\zeta \approx \pi l \sim o(L^0)$ [12] (with l as the mean free path), which gives $x_3 = 0$. As is obvious, condition (21) cannot be satisfied for any x_1, equivalently, for any $Y - Y_0$ (e.g. for any disorder conditions). As a consequence, a critical level statistics cannot occur in the one-dimensional case.

Equation (16) suggests that a 'semi-Poisson' type statistics can be seen in the $d = 1$ case for L of the order of a few mean free paths (i.e. for finite l^2/L). In the limit $L \to \infty$, however, $\Lambda_a \to 0$ and the level statistics approaches Poisson behaviour irrespective of the disorder strength.

In two dimensions, the perturbative estimate of the localization length is $\zeta \approx l \exp[\pi k_F l/2]$ with k_F as the Fermi wavenumber [12] and, in the limit $L \to \infty$, electronic states are expected to be localized even for small microscopic disorder [13]. This again corresponds to $x_3 = 0$ (thus absence of critical level statistics), and the Poisson statistics for the levels in the thermodynamic limit (as $\Lambda_a \to 0$). Note, however, that due to the exponential nature of ζ the ratio ζ^2/L can be kept non-zero and finite (by changing disorder) for a large range of L. The system can therefore show the semi-Poisson statistics in a large range of system sizes.

Case $d > 2$. For $d > 2$-dimensional, infinite systems, the change of disorder W leads to a discontinuous change in ζ and thereby Λ_a: $\zeta \propto |1 - (W/W_c)|^{-\nu}$. Here ν is the critical exponent and W_c is the critical disorder. For $W > W_c$, almost all states are exponentially localized with $\zeta \sim o(L^0)$ which results in $\Lambda_a \to 0$ and Poisson behaviour of the statistics. For $W < W_c$, the delocalization of states occurs with $\zeta \to o(L)$; this gives $\Lambda_a \to \infty$ and the Wigner–Dyson statistics. At W_c, however, the inverse participation ratio I_2 for the $d > 2$ case shows an anomalous scaling with L [14]: $I_2 \propto L^{-D_2}$ with D_2 as the multifractality exponent. This gives $\zeta^d \propto \langle I_2 \rangle^{-1} = \zeta_0^d L^{\tilde{D}_2}$ or $x_3 = \tilde{D}_2/d$ with ζ_0 as a size-independent function; note that $\tilde{D}_2 = D_2$ at the critical point [16, 17]. The size independence of the level statistics at the critical disorder therefore requires

$$D_2 = \begin{cases} d|x_1|/2 & \text{for } x_1 \leqslant -1 \\ d/2 & \text{for } x_1 > -1. \end{cases} \tag{25}$$

For example, as $x_1 \approx -1$ in case **G**, the existence of its critical point requires $D_2 \approx d/2$. Note that the numerical results for D_2, at the critical point of a $d = 3$ dimensional AE system (of type **G**) fluctuate in the range 1.4–1.6 [18, 20–22] (also see [77, 79] in [21]); this is in close agreement with the result given by equation (25) for the $d = 3$ case. The above prediction for D_2 can be used to determine the critical BE analogue for the critical state of the AE example G for the $d > 2$ case:

$$c \approx (4\pi |\alpha - \alpha_0| \zeta_0^{2d} F^2 e^{E^2})^{-1} \beta \tag{26}$$

(as $\epsilon \approx 0$). Thus, unlike the $d = 1$ case showing only Poisson level statistics in the thermodynamic limit, the energy levels of an infinite size AE for the $d > 2$ case can show three types of behaviour, namely, Poisson, Wigner–Dyson and a critical BE type statistics, at the disorder strength above, below and at the critical disorder, respectively.

The study [20] suggests a connection between D_2 and the level compressibility χ:

$$D_2 = d(1 - 2\chi). \tag{27}$$

A comparison of equation (25) with (27) gives χ for a $d > 2$-dimensional AE at the critical point: $\chi \approx 0.25$. (Note that the above χ-result is valid only for the cases of type **G** with $\zeta \propto L^{D_2/d}$.) The tail of the distribution $P(s)$ is also believed to be related to D_2 [24]: $P(\text{large } s) \approx \mathrm{e}^{-\kappa s}$ where $\kappa = (2\chi)^{-1} \approx 2$. The results for χ and κ are in close agreement with earlier numerical studies on Anderson systems [18, 20–23]; our numerical study, given in section 5, also confirms the above results. The symmetry independence of our theoretical prediction for χ and κ for Anderson systems is also in agreement with numerical observations [23, 22]. As discussed later, however, equation (27) (and therefore the above χ, κ results) seems to be valid only in the weak multifractality limit, i.e. $D_2 \sim d$ (see the paragraph below equation (36)).

For disordered systems, in general, both $Y - Y_0$ and ζ are functions of coordination number, disorder strength, hopping rate and dimensionality as well as boundary conditions of the lattice. The changing complexity due to change of the system parameters plays the role of a random perturbation, of strength $\sqrt{Y - Y_0}$, applied to the system. Here, again, the statistics of the levels is governed by Λ and, therefore, by the competition between local mean-level spacing Δ_1 and the perturbation strength $Y - Y_0$. The perturbation mixes fewer levels with increasing system size if Δ_1 increases with L at a rate faster then that of $\sqrt{Y - Y_0}$ and, as a consequence, leaves the level statistics unperturbed in the limit $L \to \infty$. In the opposite case with slower rate of change of Δ_1 with L (as compared to $\sqrt{Y - Y_0}$), even a small change in the complexity parameter is capable of mixing the levels in an increasingly large energy range of many local mean level spacings. This results in an increasing degree of eigenfunction delocalization and Wigner–Dyson behaviour of level statistics in thermodynamic limit. The critical regime occurs when both $\sqrt{Y - Y_0}$ and Δ_1 change at the same rate with L; the perturbation in this case mixes only a finite (non-zero), fixed number of levels even when the system is growing in size. As Λ remains finite in the limit $L \to \infty$, it gives rise to a new statistics different from the two end-points ($\Lambda \to 0$ and ∞). Note that the disordered systems with different dimensionalities can have different critical values of Λ (due to the dimensionality dependence of Δ_1 as well as $|Y - Y_0|$) and, therefore, correspond to critical BE analogues with different c values. Further, the boundary conditions/topologies, leading to different sparsity and coordination numbers, can also result in different critical level statistics even if the underlying symmetry and the dimensionality is the same; this is in agreement with numerical observations [18] and analytical study for 2D systems [19]. A knowledge of Λ can then be used to map the critical level statistics at the MIT for various dimensions $d > 2 \to \infty$ to the infinite family of critical BEs.

4. Determination of fluctuation measures for MIT

Many results for the spectral fluctuations of the WDT with the Poisson ensemble as an initial state are already known [4] and can be directly used for the corresponding measures for the MIT in different disordered systems.

4.1. MIT with no time-reversal symmetry

The fluctuation measures, for the Anderson transition in the presence of a magnetic field, can be given by the BEs appearing during a WDT which violates time-reversal symmetry. Such a WDT, occurring in a complex Hermitian matrix space (that is $\beta = 2$), corresponds to a transition from Poisson to GUE ensembles.

4.1.1. The two-level density correlator $R_2(r; \Lambda)$. The R_2 for BEs during Poisson $\rightarrow$ GUE transition has been obtained by various studies [4, 25, 26]. Here we use the form given in [25] for the purposes to be explained later (note that our Λ is equivalent to $\Lambda^2/2$ used in [25]),

$$R_2(r; \Lambda) = 1 + \frac{4\Lambda}{r} \int_0^\infty \mathrm{d}u \; F \mathrm{e}^{-2\Lambda u^2 - 4\pi\Lambda u} \tag{28}$$

with

$$\begin{aligned} F &= \sin(ur) f_1 - \cos(ur) f_2 \\ f_1 &= (2/z)\big[I_1(z) - \sqrt{8u/\pi} I_2(z)\big] \\ f_2 &= (1/u)\big[I_2(z) - \sqrt{2u/\pi} I_3(z)\big] \end{aligned} \tag{29}$$

where $z = \sqrt{32\pi\Lambda^2 u^3}$ and I_n as the nth Bessel function. (Note that equation (4.15) in [25] has a misprint in the coefficient of u in the exponent; the correct coefficient is given in equation (28) above.)

Equation (28) gives the exact form of two-point correlation for the Anderson transition with no time-reversal symmetry. Here $R_2(r, \infty) = 1 - (\sin^2(\pi r)/\pi^2 r^2)$ and $R_2(r, 0) = 1$ corresponding to the metal and insulator regimes respectively. A substitution of the critical value of Λ_a in equation (28) will thus give R_2 for the critical AE.

For large Λ values (for all r), R_2 can be approximated as follows [25, 26]: $R_2 = 1 - Y_2$ where

$$\begin{aligned} Y_2(r, \Lambda) &= \frac{-4\Lambda}{16\pi^2\Lambda^2 + r^2} - \frac{1}{2\pi^2 r^2}[\cos(2\pi r)\mathrm{e}^{-\frac{r^2}{2\Lambda}} - 1] \\ &\approx \frac{3}{2\pi^2\Lambda} \frac{\sin^2(\pi r)}{\sinh^2(r\sqrt{3/2\Lambda})} \qquad (\text{for } r \ll \sqrt{\Lambda}). \end{aligned} \tag{30}$$

However, for $r > \sqrt{\Lambda}$, $Y_2(r, \Lambda) = -\frac{4\Lambda}{16\pi^2\Lambda^2 + r^2} + \frac{1}{2\pi^2 r^2}$. As $\Lambda = (4c\pi\gamma)^{-1}$ (near $E = 0$) for a critical BE, $Y_2 \approx (1 - 8\pi^2\Lambda)/2\pi^2 r^2$ for $r > 2\beta\pi\Lambda$ (here $\beta = 2$).

The above large r behaviour of $Y_2(r; \Lambda)$ at $\Lambda = \Lambda_b$ results in a non-zero, fractional value of the sum $I = \int_{-\infty}^{\infty} Y_2(r; \Lambda)\mathrm{d}r$ for a critical BE of complex-Hermitian type:

$$I \approx 1 - (\beta\pi^2\Lambda)^{-1}. \tag{31}$$

Note that a $0 < I < 1$ value is believed to be an indicator of the multifractality of the wavefunctions and the fractional compressibility of the spectrum ($I = 1, 0$ for the WD and Poisson cases, respectively) [20, 27]. A fractional behaviour of I and the multifractality is already known to exist in critical AE [20, 27]. Using $\Lambda_a = \Lambda_b$ in equation (31), one can now determine the measure I for an AE: $I \approx 1 - (\beta\pi^2\Lambda_a)^{-1}$.

4.1.2. Nearest neighbour spacing distribution P(S). The nearest neighbour spacing distribution $P(s)$ for the MIT with no time-reversal symmetry can similarly be given by using the one for the BE during the Poisson $\rightarrow$ GUE transition [28, 29]:

$$P(s; \Lambda) \propto \frac{s}{\sqrt{2\pi\Lambda}} \mathrm{e}^{-s^2/8\Lambda} \int_0^\infty \mathrm{d}x \; \mathrm{e}^{-x - x^2/8\Lambda} \frac{\sinh(xs/4\Lambda)}{x}. \tag{32}$$

A substitution of $\Lambda \to \infty$ and $\Lambda \to 0$ in the above equation gives the correct asymptotic limits, namely, Wigner–Dyson and Poisson, respectively: $P(s;\infty) = P_{\rm w}(s) = 32s^2 {\rm e}^{-4s^2/\pi}/\pi^2$ (WD limit) and $P(s;0) = P_{\rm p}(s) \propto {\rm e}^{-s}$ (Poisson limit). (Although this result is rigorous for a 2×2 matrix space it is proved reliable for systems with many levels; see [29].)

4.1.3. Level compressibility. The level compressibility $\chi = 1 - \int_{\infty}^{\infty} Y_2(r)\,{\rm d}r = 1 - I$ is an important characteristic of the critical level statistics and the multifractal nature of the wavefunctions.

χ for a BE can be obtained by using equation (28),

$$\chi(\Lambda) = 1 - 4\pi\Lambda \int_0^{\infty} {\rm d}u\ f_1(z) \exp[-2\Lambda u^2 - 4\pi\Lambda u] \tag{33}$$

$$\approx 1 - 4\pi^2\Lambda \qquad \text{for small } \Lambda \tag{34}$$

$$\approx (2\pi^2\Lambda)^{-1} \qquad \text{for large } \Lambda. \tag{35}$$

The substitution of equation (16) for Λ_b in equations (29) and (30) gives, in the band around $E = 0$, $\chi = 1 - (\pi/\gamma c)$ and $\chi \approx (4\gamma c/2\pi)$, in small and large Λ limits, respectively. Thus a critical BE characterized by a finite c value shows a fractional level compressibility. As is clear from the above, $\chi \to 1$ for $\Lambda \to 0$ (or $c \to \infty$) which corresponds to a Poisson behaviour, and $\chi \to 0$ for $c \to 0$ or $\Lambda \to \infty$ which corresponds to the GUE statistics.

The compressibility of the energy levels of Anderson systems at their critical point is already known to be fractional, with $\chi = 0, 1$ in the metallic and the insulator phase, respectively. The existence of a fractional χ for both critical BE and critical AE is consistent with our claim about their spectral analogy. The compressibility of the AE with different types of disorders and lattices can now be obtained just by finding the same for their critical BE analogues.

For the critical BE case ($d = 1$) with large Λ_b (equivalently, small c), equation (27) along with equation (35) gives

$$D_2 = 1 - 4\gamma c/\pi \qquad \text{for small } c. \tag{36}$$

Equation (36) gives the correct fractal dimension in the limit $c \to 0$ (the Wigner–Dyson limit): $D_2(c = 0) = 1$. However, for small Λ_b (or large c), equation (27) implies $D_2 = 2\pi/\gamma c - 1$ and therefore $D_2 = -1$ in the Poisson limit $c \to \infty$, which is different from the expected result $D_2 = 0$ for the localized states. As mentioned in [16], similar violation of equation (27) is indicated by numerical data for the tight-binding models in dimensions $d > 4$. The observed inaccuracy of equation (27), for both AE and BE in the strong multifractality limit, also lends credence to our claim regarding AE–BE analogy.

It is worth mentioning here that, similar to the AE–BE mapping, the spectral statistics of any generalized Gaussian ensemble, with probability density given by (1), can be mapped to BEs [2]. The non-validity of equation (27) for critical BEs with small c parameters implies, therefore, the same violation for all generalized Gaussian ensembles in the strong multifractality limit. The implication is already known to be correct for the power law random banded matrices [16, 17, 33] (also see section 6) and for the random matrix ensemble introduced by Moshe, Neuberger and Shapiro (later referred to as the MNS model) [32].

4.2. MIT with time-reversal symmetry

The statistical measures for the Anderson transition in the presence of a time-reversal symmetry can similarly be obtained by using their equivalence to a WDT preserving the same symmetry, that is, a transition from Poisson to GOE ensembles; the latter occurs in a real-symmetric matrix

space (here $\beta = 1$). However, due to the technical difficulties [4], only some approximate results are known for the latter case.

4.2.1. The two-level density correlator $R_2(r; \Lambda)$. The R_2 for small r can be obtained by solving equation (17) of [2] for $\beta = 1$ which gives

$$R_2(r, \Lambda) \approx (\pi/8\Lambda)^{1/2} r \mathrm{e}^{-r^2/16\Lambda} I_0(r^2/16\Lambda) \tag{37}$$

with I_0 as the Bessel function.

Similarly, for large-r behaviour, R_2 can be shown to satisfy the relation (see equation (23) of [4])

$$R_2(r, \Lambda) \approx R_2(r, \infty) + 2\beta\Lambda \int_{-\infty}^{\infty} \mathrm{d}s \, \frac{R_2(r-s; 0) - R_2(r-s; \infty)}{(s^2 + 4\pi^2\beta^2\Lambda^2)} \tag{38}$$

$$\approx R_2(r, \infty) + 2\beta\Lambda/(r^2 + 4\pi^2\beta^2\Lambda^2) \tag{39}$$

where $\beta = 1$ and $R_2(r, \infty) = 1 - \sin^2(\pi r)/\pi^2 r^2 - \left(\int_r^{\infty} \mathrm{d}x \, \sin \pi x/\pi x\right) \left(\frac{\mathrm{d}}{\mathrm{d}r} \sin \pi r/\pi r\right)$ (GOE limit).

As can be seen from the above, $Y_2(r, \Lambda) \approx -\frac{2\Lambda}{4\pi^2\Lambda^2 + r^2} + \frac{1}{2\pi^2 r^2}$ for $r > \sqrt{\Lambda}$. However, note that, for $r > 2\pi\Lambda$, the behaviour of Y_2 is different from that of a BE with no TRS: $Y_2 \approx (1 - 4\beta\pi^2\Lambda)/2\pi^2 r^2$. This further suggests the following behaviour of I: $I = 1 - (\beta\pi^2\Lambda)^{-1}$. The I for a critical BE is therefore symmetry dependent (as $\Lambda = \Lambda_b$ does not depend on β). However, the I for its AE analogue is independent of the symmetry parameter β; this is because $\Lambda = \Lambda_a \propto \beta^{-1}$ in this case (see equation (19)).

4.2.2. Nearest neighbour spacing distribution P(S). $P(s)$ for this case can be given by using the one for a BE during the Poisson $\rightarrow$ GOE transition [28]:

$$P(s, \Lambda) = (\pi/8\Lambda)^{1/2} s \mathrm{e}^{-s^2/16\Lambda} I_0(s^2/16\Lambda) \tag{40}$$

with I_0 as the Bessel function; note that, as expected, this is the same as the R_2 behaviour for small r (equation (37)).

4.2.3. Level compressibility. The lack of the knowledge of $R_2(r, \Lambda)$ for the entire energy range handicaps us in providing an exact form of the compressibility for the time-reversal case. However, its approximate behaviour can be obtained by using the relation $\chi = 1 - I$. Thus, for a time-reversal critical BE ($\beta = 1$),

$$\chi \approx (\pi^2\Lambda)^{-1} \tag{41}$$

and therefore $\chi \approx (\pi^2\Lambda_a)^{-1}$ for its AE analogue.

Equations (35), (41) indicate the influence of underlying symmetry on the compressibility of the levels: $\chi \approx (\beta\pi^2\Lambda)^{-1}$. Note that χ for a BE is symmetry dependent due to $\Lambda = \Lambda_b$ being β independent (see equation (16)). However, as $\Lambda = \Lambda_a \propto \beta^{-1}$ for an AE (see equation (19)), its χ would be symmetry independent; this is in agreement with numerical observations for Anderson systems [23, 22, 34]. This further implies that the critical BEs corresponding to critical AEs with and without time-reversal symmetry would be different.

In the past, an attempt to explain the symmetry independence of the level statistics at the Anderson transition was made in the study [34] by suggesting a scaling behaviour of the distribution $P(s)$ with the conductance g and the symmetry parameter β. The $P(s)$ in the study [34] was obtained by interpolation between metallic and insulator limits. We note that, by using the Λ–g connection (equation (22)), the $P(s)$ given by equations (32) and (40) can also be expressed as a function of g.

5. Numerical comparison of the level statistics of critical AE and critical BE

In this section, we investigate the AE–BE spectral analogy by numerically comparing two of their fluctuation measures, namely, $P(s)$ and the number variance $\Sigma^2(r) = \langle (r - \langle r \rangle)^2 \rangle$. The former is a measure of the short range correlations in the spectrum and the latter, describing the variance in the number of levels in an interval of r mean level spacings, contains the information about the long range correlations [3]. Σ_2 is also an indicator of the compressibility of the spectrum; $\lim r \to \infty\ \Sigma_2(r) \approx \chi r$. To study the AE–BE analogy in the presence of time-reversal symmetry as well as its absence, we consider two cases of the three-dimensional AE (simple cubic lattice of size $L = 13$ and with Gaussian site disorder) in the critical regime.

(i) AE_t. The AE with isotropic random hopping, hard wall boundary conditions and time-reversal symmetry; here $W = 4.05$, $W_1 = 1$, $W_2 = 0$, $t_1 = 0$, $t_2 = 0$. The criticality of the AE for the same disorder parameter values but with periodic boundary conditions is numerically studied in [30]. However, the system remains in the critical regime under hard wall boundary conditions too.

(ii) AE_{nt}. The system **G** with isotropic non-random hopping, periodic boundary conditions and no time-reversal symmetry; here $W = 21.3$, $W_1 = 0$, $W_2 = 0$, $t_1 = 1$, $t_2 = 1$. The time-reversal symmetry is broken by applying an Aharonov–Bohm flux ϕ which gives rise to a nearest neighbour hopping $H_{kl} = \exp(\mathrm{i}\phi)$ for all k, l values related to the nearest neighbour pairs [22]. The flux ϕ is chosen to be non-random in nature, that is, $\langle \cos^2(\phi) \rangle = W_1 = 0$, $\langle \sin^2(\phi) \rangle = W_2 = 0$ and $\langle \cos(\phi) \rangle = t_1 = 1$, $\langle \sin(\phi) \rangle = t_2 = 1$.

We study each AE case for two system sizes $L = 10$ and 13 by numerically diagonalizing the matrices of the ensembles by standard techniques. Each ensemble contains a few thousand matrices and the statistical average is performed approximately over 3×10^5 levels, obtained by taking 200 levels in a small energy range around the centre $E = 0$ of the spectrum of each matrix. Each BE (chosen with $\gamma = 2$) is also analysed for two dimensions $N = 1000$ and 3000. Note that due to a higher rate of change of the mean level density, the Λ (equation (16)) for BEs changes more rapidly with energy as compared to AE cases (equation (19)). To avoid mixing of levels with different transition rates, therefore, fewer ($\approx$100) levels are taken from the spectrum of a matrix in the BE case; the total number of levels for BE analysis is kept nearly the same (as in the AE cases) by taking a bigger ensemble. Each spectrum is unfolded for $P(s)$ and $\Sigma_2(r)$-analysis. The unfolding is carried out by numerically calculating the unfolded levels $r_j = \int_{-\infty}^{E_j} \mathrm{d}x\, R_1(x)$ for $(j = 1, 2, \ldots, N)$ with symbol E_j used for levels before unfolding.

The parameter $Y - Y_0$ for both AE cases is given by equation (5). As $N|Y - Y_0| \simeq \mathrm{o}(1)$, the mean level density R_1 for both AE cases is given by equation (14); $R_1 = NF$ with F as an N-independent function of energy. This is confirmed by our numerical analysis of R_1 for different N values for each AE case, with figure 2(a) showing the comparison for only two N values. (Note that for R_1 study the spectrum is not unfolded and almost all the eigenvalues of each matrix are used for the analysis.) As mentioned in section 3, the critical BE analogues for the fluctuation measures need not have the same R_1. The R_1 behaviour for the critical BE analogues for the fluctuations of the AE cases is shown in figure 2(b); the numerical fitting confirms that $R_1 = (N/\sqrt{\pi})\mathrm{e}^{-E^2}$ for each critical BE case which is quite different from their AE analogues.

Figures 3 and 4 show the $P(S)$ and $\Sigma_2(r)/r$ behaviour for the two AE cases. The almost the same behaviour for two system sizes in each AE case, for both the measures, confirms their critical nature. We find, from figure 4, that the large r behaviour of $\Sigma^2(r)/r$ for both AEs seems to converge to $\chi \approx 0.25$ which confirms our analytical prediction of χ for AEs (with

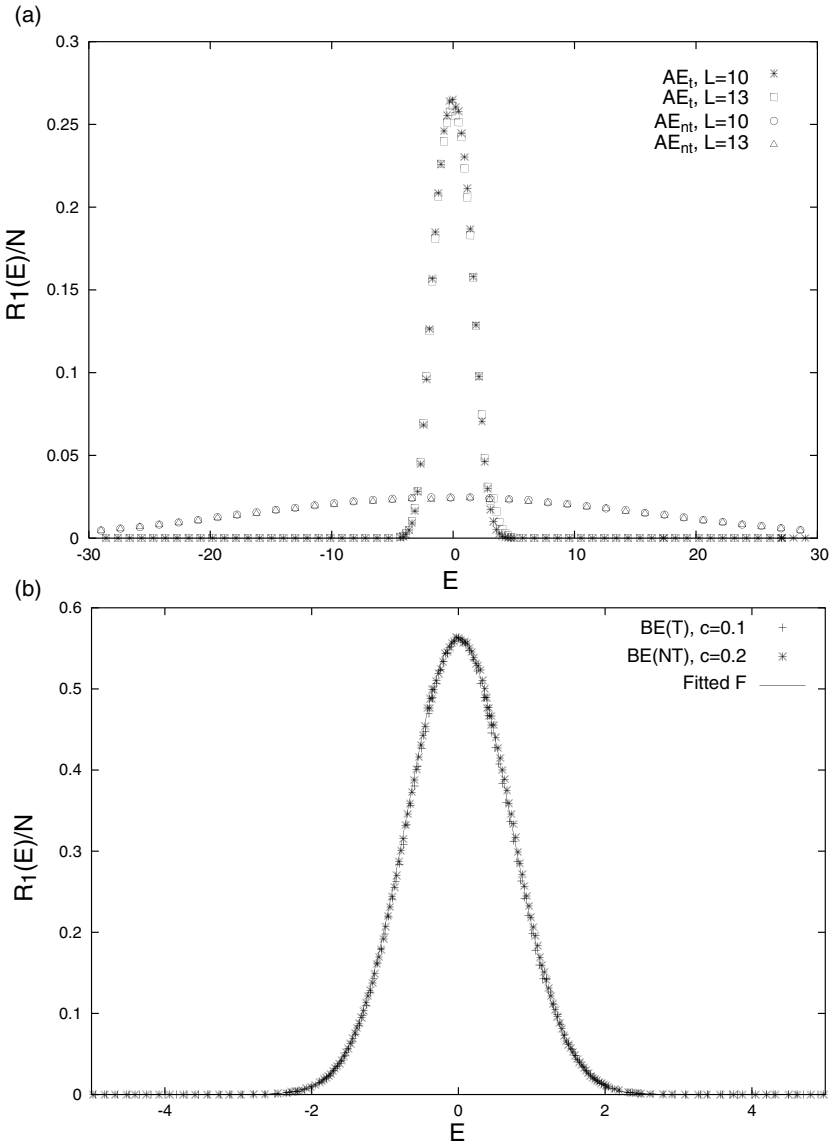

Figure 2. The behaviour of level density $F(E) = N^{-1} \cdot R_1$. (a) For the two cases for two system sizes $L = 10$ and 13. The numerical fitted function has the form $F = f_1 e^{-(E^2/f_2)}$ with $f_1 = 0.16$, $f_2 = 5$ for AE_t and $f_1 = 0.016$, $f_2 = 400$ for AE_{nt}. (b) For the critical BE analogues of the higher order correlations of the two cases considered in (a). Here F for all the critical BE cases is well fitted by the function $F(E) = \pi^{-1/2} e^{-E^2}$. Note the lack of analogy between the mean level densities for the cases given in (a) and (b) while their higher order correlations (shown in figures 3, 4) are approximately the same.

$\zeta \sim o(L^{D_2/d})$, based on equality of equations (25) and (27)); it is also in agreement with other numerical studies [18, 20–22].

The determination of the critical BE analogue of the fluctuation measures of an AE requires a prior information about Λ_a (given by equation (19)). Although we know the function F for each AE case (see figure 2(a)) as well as $\alpha - \alpha_0$ (from equation (6), $\alpha - \alpha_0 = 1.36, 5.43$ for AE_t

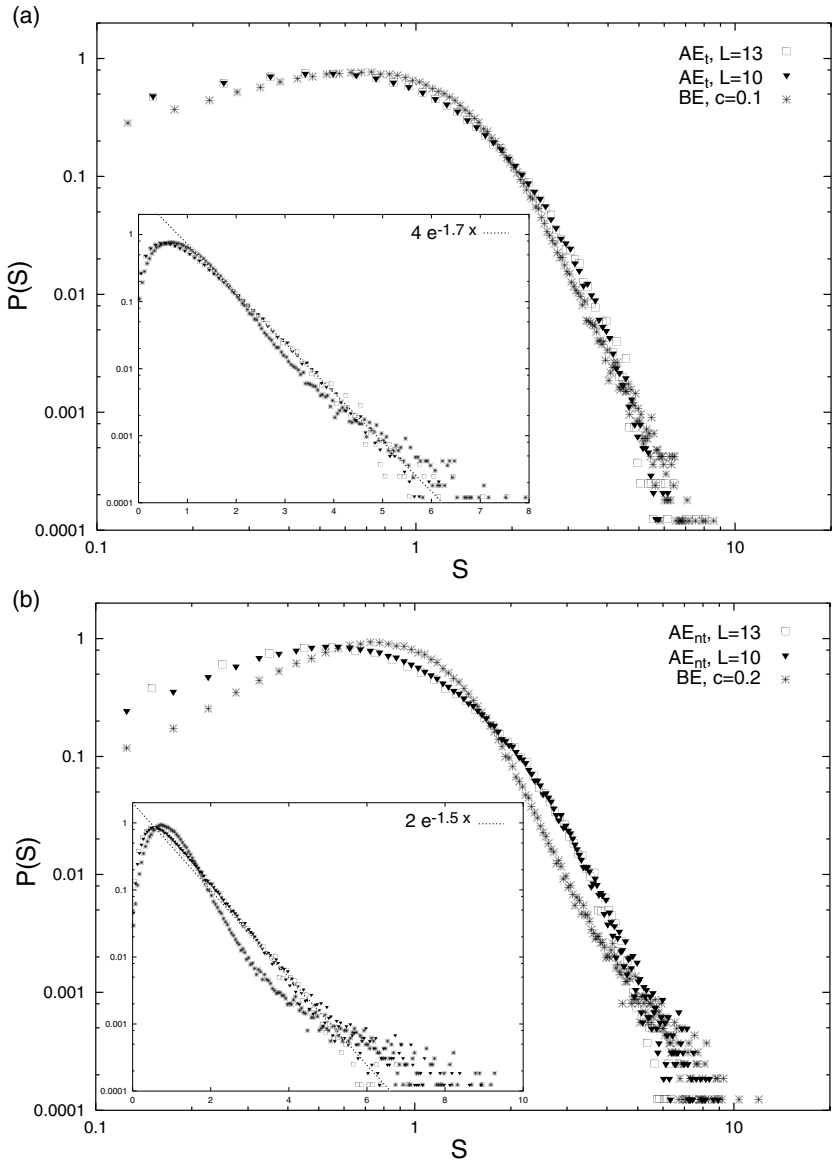

Figure 3. The comparison of distribution $P(S)$ of the nearest neighbour spacings S for the AE ($d = 3$) and BE cases on a log–log scale. To confirm the critical state of the AE, the distribution is shown for two system sizes L for each AE case. The insets show the same functions on a lin–log scale and also compare the behaviour with $e^{-\kappa s}$. (a) AE$_t$ (with hard wall boundary conditions, random hopping and time-reversal symmetry) and its critical BE analogue ($c = 0.1$). The dashed line in the inset is the fitted function $f = 4e^{-1.7S}$ which gives $\kappa \approx 1.7$. (b) AE$_{nt}$ (with periodic boundary conditions, non-random hopping and no time-reversal symmetry) along with its critical BE analogue $c = 0.2$. The dashed line in the inset is the fitted function $f = 2e^{-1.5S}$ which gives $\kappa \approx 1.5$.

and AE$_{nt}$, respectively), the determination of ζ requires an statistical analysis of wavefunctions. Fortunately, equations (35) and (41) suggest that the parameter c of the critical BE analogue of a critical AE can also be obtained (approximately) from its χ behaviour: $c \approx (\beta\pi\chi/4\gamma)$.

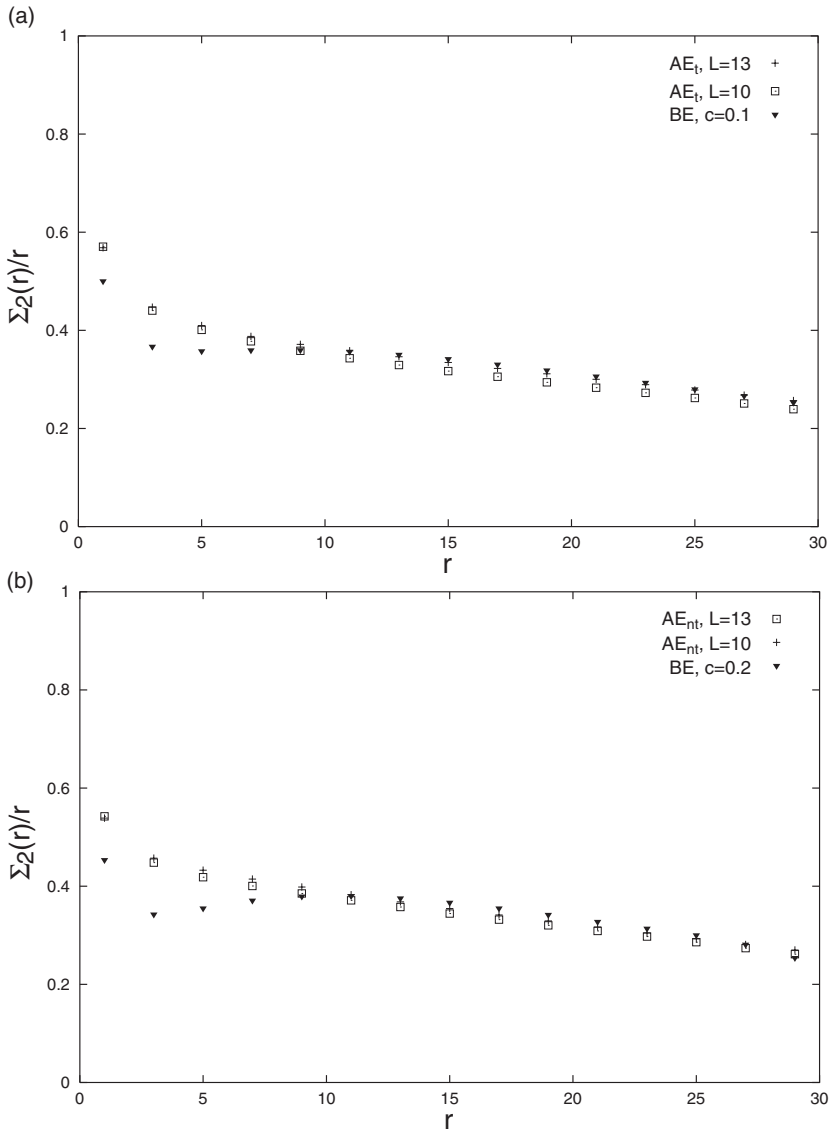

Figure 4. The comparison of the $\Sigma^2(r)/r$ behaviour for the AE and BE cases. (a) AE_t and the corresponding critical BE analogue ($c = 0.1$). (b) AE_{nt} and the corresponding critical BE ($c = 0.2$). Note here the critical BE analogue for each AE case is same as for the $P(S)$ study. As can be seen, Σ^2/r for large r seems to approach the limit suggested by the relation $\chi = \lim_{r\to\infty} \Sigma^2/r = (4\gamma c/\beta\pi)$, that is, $\chi = 0.25$. Note that this is the expected χ for AEs on the basis of equations (25) and (27) too. Besides showing AE–BE analogy, the figure also confirms (i) the symmetry independence of χ for AEs and (ii) the fractional χ result for a critical BE.

Using $\chi \approx 0.25$, the theoretically expected c parameters for AE_t and AE_{nt} are 0.1 and 0.2, respectively. Figures 3 and 4 confirm the existence of the critical BE analogues, of the AE cases, at the above c values. Note that the above relation between the parameter c of a critical BE and χ of a critical AE is obtained by combining the theoretical results for (i) critical point D_2 behaviour predicted by our Λ formulation, (ii) D_2 given by equation (27), (iii) χ for a BE

(equations (35) and (41)) and (iv) AE–BE analogy. The good AE–BE agreement observed in figures 3 and 4 therefore indicates the validity of all the above formulations used to derive the c(BE)–χ(AE) relation.

The exponential decay of the tail of $P(s)$ for the $d > 2$ AE system at the critical point has been confirmed by various numerical studies (for example, see [7, 18, 20–22, 31]). The validity of the AE–BE analogy requires a similar decay of $P(s)$ for the critical BEs too. Figure 1(b) compares $P(s)$ behaviour for a few critical BEs with the function $e^{-\kappa s}$. The fitted κ values are close to $\kappa \approx (\beta\pi/8\gamma c)$ for intermediate c ranges; the κ–c relation is obtained by using $\kappa = (2\chi)^{-1}$ and $\chi = (4\gamma c/\beta\pi)$ at $E = 0$ (see below equations (35)). The insets in figure 3 compare the tails of the $P(s)$ for the AEs and their BE analogues with function $e^{-\kappa s}$; we find $\kappa \approx 1.5$–1.7. The result is close to our analytical prediction $\kappa \approx 2$ for the critical AE case (see below equation (27)). However, the lack of exact agreement seems to suggest the approximate nature of the χ–κ relation, namely, $\kappa = (1/2\chi)$ (note that as our analytical prediction $\chi = 4\gamma c/\beta\pi$ is found to be in excellent agreement with the numerics, this leaves the χ–κ relation as the possible source of error).

The study [27] claims that the critical level statistics in the Rosenzweig–Porter ensemble (similar to BE, as mentioned in section 2) does not have a fractional compressibility and, therefore, is different in nature from that for critical AE. However, our analytical results, supported by the numerical evidence, disprove their claim. Our numerical study confirms the existence of a fractional χ, increasing with c, for various critical BEs. Two such cases are shown in figure 4, with their χ results in close agreement with our analytical prediction, namely, equations (35) and (41).

6. Connection with PRBM model

In past, a random matrix ensemble, namely the power law random banded matrix (PRBM) ensemble, was suggested as a possible model for the critical level statistics of the Anderson Hamiltonian [6]. A PRBM ensemble is defined as the ensemble of random Hermitian matrices with matrix elements H_{ij} as independently distributed Gaussian variables with zero mean, i.e. $\langle H_{ij}\rangle = 0$, and variance $\langle H^2_{ij;s}\rangle = G^{-1}_{ij}\left[1 + (|i-j|/b)^2\right]^{-1}$, $G_{ij} = \beta(2 - \delta_{ij})$ and $G_{ij} = 1/2$. It is critical at arbitrary values of the parameter b and is believed to show all the key features of the Anderson critical point, including multifractality of eigenfunctions and the fractional spectral compressibility.

The success of the PRBM ensemble, a one-dimensional system, as a model for Anderson systems in arbitrary dimension is a little surprising. However, it can be explained on the basis of our formulation. The PRBM–AE connection is a special case of our study connecting any generalized Gaussian ensemble with the BE. The PRBM ensemble being Gaussian in nature, its complexity parameter can be defined by using equation (4) which can then be used to obtain its BE analogue. Equation (4) gives (with $\gamma = \beta$)

$$Y - Y_0 = \frac{2}{N(N+2-\beta)} \sum_{r=1}^{N} (N-r)\ln|1 + (b/r)^2|^2 \tag{42}$$

which gives $Y - Y_0 \propto f(b)/N$ with $f(b) \approx 2b^{0.85}\ln(5b)$ for $b \gg 1$ and $f(b) \approx 2b^{1.75}$ for $b \ll 1$. As $Y - Y_0 \approx o(1/N)$, R_1 can then be given by equation (14). Following equation (15), Λ for a PRBM ensemble is

$$\Lambda_{\rm prbm}(b, E) = f(b)F^2(E)\zeta^2 N^{-1}. \tag{43}$$

The well known size independence of level statistics for the PRBM case for all b values requires $\Lambda_{\rm prbm}$ to be N independent, which gives $\zeta \propto N^{1/2}$ for all b ranges by our formulation.

However, note that, for the PRBM model, $\langle I_2\rangle \propto N^{-D_2}$ with D_2 as a function of b [16, 17, 33]. The use of $\zeta \propto \langle I_2\rangle^{-1}$ in equation (43), therefore, gives a size-dependent $\Lambda_{\rm prbm}$. Keeping in view the well known criticality of the PRBM system for all b ranges, it seems that the relation $\zeta \propto \langle I_2\rangle^{-1}$ is not valid for the PRBM case.

Using the prediction $\zeta = \zeta_0 N^{1/2}$ in equation (43) and the relation $\Lambda_{\rm prbm} = \Lambda_b$, the level statistics of a PRBM can be mapped to a critical BE ensemble with

$$c = (4\pi\beta\Lambda_{\rm prbm})^{-1}\mathrm{e}^{-E^2} = (4\pi\beta f(b)\zeta_0^2 F^2(E)\mathrm{e}^{E^2})^{-1}. \tag{44}$$

The spectral statistics of the PRBM therefore shows a crossover from Poisson (as $c \to \infty$ for $\Lambda_{\rm prbm} \to 0$ i.e. $b \to 0$) to Wigner–Dyson behaviour ($c \to 0$ for $\Lambda_{\rm prbm} \to \infty$ or $b \to \infty$).

The spectral compressibility χ for a PRBM ensemble at $E = 0$ can now be obtained by substituting $\Lambda = \Lambda_{\rm prbm} = f(b)\zeta_0^2$ in equations (34), (35) and (41) which give

$$\chi = \begin{cases} 1 - 4\pi^2\zeta_0^2 f(b) & \text{for } b \ll 1 \\ (\beta\pi^2\zeta_0^2 f(b))^{-1} & \text{for } b \gg 1. \end{cases} \tag{45}$$

The above results are at least in the same form as obtained in [16, 17, 33]; the lack of explicit knowledge of ζ_0 prevents us from making any further comparison. As χ, in equation (45), changes from zero to unity with decreasing b, it violates the relation (27) in the range $b \ll 1$. The same violation was observed in previous PRBM studies [16] too. Thus our results obtained by using the PRBM-BE analogy seem to be in accordance with earlier studies on the PRBM model.

In brief, the PRBM ensemble, with b as a parameter, can be mapped to the critical BE with parameter c (see equation (44)). As a consequence, the studies suggesting the analogy of spectral statistics for the PRBM and AE ensembles are in good agreement with our study claiming the AE–BE analogy. By using the connection of PRBM with the MNS model [27, 32], the PRBM–BE–AE analogy can further be extended to the MNS–BE–AE analogy.

7. Conclusion

Finally, we re-emphasize our main result.

Under the independent electron approximation, the level statistics for the disordered systems undergoing a localization $\to$ delocalization transition of wavefunctions can be described by the Brownian ensembles (with uncorrelated elements) undergoing a similar transition.

The analogy helps us in making the following deductions.

(i) The transition in the statistics is governed by a single scaling parameter $\Lambda = \frac{|Y-Y_0|}{\Delta^2}(\frac{\zeta}{L})^{2d} = f(\frac{\zeta}{L})$. The second equality follows from the dependence of wavefunction statistics e.g. inverse participation ratio I_2 and therefore ζ on the complexity parameter $|Y - Y_0|$.

(ii) The level statistics is governed by the competition between the complexity parameter and the local mean level spacing. The critical point of level statistics occurs when the complexity parameter $Y - Y_0$ and Δ_1 have the same size dependence. In particular, if $|Y - Y_0| \sim \mathrm{o}(N^\alpha)$ and $\Delta_1 \sim \mathrm{o}(N^\beta)$ for a disordered system, its critical point will occur when $\alpha - 2\beta = 0$. However, if the local mean level spacing in the system changes at a faster rate with size as compared to $\sqrt{Y - Y_0}$ (i.e. $\beta > \alpha/2$), the system will never reach its critical state and will always remain in the localized regime.

(iii) The critical BE analogue of a critical AE is unique. Further, it is different for a critical AE with and without time-reversal symmetry. Similar to AEs, the level statistics of BEs shows a scaling behaviour as well as a critical point with fractional level compressibility.

However, unlike AEs, χ turns out to be symmetry dependent for BEs, their parameter Λ being symmetry independent.

(iv) The AE–BE analogy confirms the symmetry independence of the compressibility of levels and the multifractality of the wavefunctions at the critical point of the Anderson transition. The analogy also indicates the non-validity of the relation $D_2 = d(1 - 2\chi)$ in the strong multifractality limit, and the approximate nature of the relation $\kappa = (2\chi)^{-1}$.

(v) The AE–BE analogy helps us in formulating, for the first time, the exact two-point level density correlation at the critical point of a disordered system. The formulation is applicable for a wide range of system parametric conditions.

It should be noted that both MIT as well as WDT occur due to delocalization of the wavefunctions. In fact, our analytical study suggests that the level statistics of almost all complex systems undergoing a localization $\rightarrow$ delocalization transition follows the same route although with different transition rates; the state of level statistics of two systems with different complexity may correspond to two different points on this route. In principle, our analytical work is applicable to the Gaussian models of complex systems only; however, the intuition based on earlier studies suggests the validity of the results for the systems with other origins of randomness too [3]. For example, the investigation of a number of dynamical systems seems to support this intuition. It has been shown that the spectral statistics of pseudo-integrable billiards is remarkably similar to the critical statistics of AE [35]. The presence of a statistics intermediate between the Poisson and GOE has already been shown for the kicked rotor in the non-integrable regime of the kicking parameter [36]. A correspondence of the integrable systems to the insulators and of the chaotic systems to the metals is already known to exist. The integrability $\rightarrow$ chaos transition in the dynamical systems therefore seems to follow a route in the level statistics similar to that of the MIT; note that such a transition in classical systems corresponds to a delocalization of the wavefunctions in their quantum analogue. Thus the analogy of the statistical level fluctuations between the AE and BE may possibly be extended to dynamical systems and the BE too; if the latter is found to be correct, the analogy would be useful to obtain the correlations for the non-integrable regime.

The evidence of such an analogy would suggest the existence of several features, unknown so far, for the level statistics of dynamical systems. For example, the analogy can be used to intuitively claim and search for the existence of a critical point, the dimensional dependence of level statistics and the multifractality of eigenfunctions during the transition from Integrable to chaotic dynamics. It should be noted that a generic one-dimensional dynamical system always shows a Poisson level statistics (in analogy with one-dimensional AE). However, the dynamics in a three-dimensional system shows a feature called 'Arnold diffusion', absent in lower dimensions. The intuition based on the above analogy suggests the possible existence of a critical level statistics at the parametric values at which Arnold diffusion takes place. A further exploration of such an analogy is therefore highly desirable.

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